Perturbation-Theoretical Integral Representation and the High-Energy Behavior of the Scattering Amplitude. II*

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It is established with almost complete generality by means of the perturbation-theoretical integral representation that the high-energy behavior of the exact scattering amplitude for spinless particles is closely related to the continued partial-wave Bethe-Salpeter equation, namely, the latter determines the generalized Regge trajectory equation. The Regge behavior is derived for the exact scattering amplitude of two identical neutral spinless particles.

1. INTRODUCTION

RECENTLY, Bertocchi, Fubini, and Tonin,[1] and others[2,3] have investigated the high-energy behavior of the scattering amplitude on the basis of the multiperipheral model. They have argued that it will exhibit the Regge behavior and that the Regge trajectory is determined by the Bethe-Salpeter equation. The present author[4] has pointed out that equivalent results can be obtained more elegantly by means of the perturbation-theoretical integral representation,[5] but the Regge behavior is merely an ansatz and not proven mathematically. The possible high-energy behavior in the t channel has been shown to be proportional to

$$(-t)^l[\ln(-t)]^n[\ln\ln(-t)]^r\cdots, \qquad (1.1)$$

where l is a function of s which is determined by the partial-wave Bethe-Salpeter equation.

All the above arguments concern the ladder approximation. This approximation is the simplest one but it badly violates the crossing symmetry properties. Hence one may wonder to what extent the above results remain valid. The purpose of the present paper is to answer this question. It should be remarked that the integral equation in the multiperipheral model cannot be extended to the general nonladder-like graphs. Hence we shall employ the perturbation-theoretical integral representation.[5] It is much more reliable than the Mandelstam representation which is widely assumed, because the former can be proven in every finite order of perturbation theory. Our present analysis definitely shows that *the high-energy behavior mentioned above is almost completely general* as far as spinless particles are concerned. Oehme[6] has shown that l in (1.1) with $\mathrm{Re}\,n>-1$ cannot be a constant ($\mathrm{Re}\,l>-\frac{1}{2}$) independent of s (apart from the possible contributions from "elementary particles" in the s channel) under certain assumptions.[7] Our conclusion also supports that l should not be independent of s, because it is very unlikely that the Bethe-Salpeter equation might have a constant eigenvalue.

The next section is devoted to the explanation of the notations used. In Sec. 3, we derive the integral equations for weight functions. In Sec. 4, their support properties are discussed in general mass case. Sections 5 and 6 deal with the high-energy behavior of the scattering amplitude. We generally establish its connection with the partial-wave Bethe-Salpeter equation. In the final section, we discuss a nonlinear integral equation for elastic scattering amplitudes, and derive the Regge behavior in the case of identical-neutral-particle scattering.

2. PRELIMINARIES

We consider an inelastic scattering $A+B\to C+D$, where A, B, C, D stand for spinless particles, whose masses are denoted by m_A, m_B, m_C, m_D, respectively. Let $2k$, q, and p be the total momentum, the relative momentum in the initial state, and that in the final state, respectively. The initial momenta $k+q$ and $k-q$ lie on the mass shells. Then the integral equation for the Feynman amplitude of our process can generally be written as

$$f(p,q)=g(p,q)+\int d^4p'G(p,p')f(p',q). \qquad (2.1)$$

Here $f(p,q)$ stands for the Feynman amplitude in question, $g(p,q)$ being the contribution from the Feynman graphs which contain no $C+D$ intermediate states. $G(p,p')$ denotes the irreducible Green function for $C+D\to C+D$ apart from the two one-particle propagators. The constant momentum k has been suppressed in the arguments for simplicity.

We also consider the corresponding partial-wave Bethe-Salpeter equation in the rest frame ($\mathbf{k}=0$):

$$\mathcal{Y}_{lm}(\mathbf{p})f_l(p)=\int d^4p'G(p,p')\mathcal{Y}_{lm}(\mathbf{p}')f_l(p'), \qquad (2.2)$$

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[1] L. Bertocchi, S. Fubini, and M. Tonin, Nuovo Cimento **25**, 626 (1962).

[2] C. Ceolin, F. Duimio, R. Stroffolini, and S. Fubini, Nuovo Cimento **26**, 247 (1962).

[3] D. Amati, A. Stanghellini, and S. Fubini, Nuovo Cimento **26**, 896 (1962).

[4] N. Nakanishi, Phys. Rev. **133**, B214 (1964). This paper is referred to as I.

[5] N. Nakanishi, Progr. Theoret. Phys. (Kyoto) **26**, 337 (1961); *ibid.*, Errata **28**, 406 (1962).

[6] R. Oehme, Phys. Rev. Letters **9**, 358 (1962).

[7] His proof needs a tacit assumption in addition to the Mandelstam representation and the absence of natural boundaries. In order to get the continued unitary condition, the boundary values $F_\pm(s\pm i0,\lambda)$ must be assumed to be analytic also in $-\frac{1}{2}<\mathrm{Re}\lambda\leq N$.

where $\mathcal{Y}_{lm}(\mathbf{p})$ stands for a solid harmonic, $f_l(p)$ the partial-wave Bethe-Salpeter amplitude.

The invariant squares which we use are as follows:

$$\begin{array}{ll} (k+q)^2=m_A^2\,, & (k-q)^2=m_B^2\,, \\ (k+p)^2=v\,, & (k-p)^2=w\,, \\ (k+p')^2=v'\,, & (k-p')^2=w'\,, \\ (p-q)^2=t^{(0)}\equiv t\,, & (p+q)^2=t^{(1)}\equiv u\,, \\ (p'-q)^2=t^{(0)\prime}\equiv t'\,, & (p'+q)^2=t^{(1)\prime}\equiv u'\,, \\ (p-p')^2=r^{(0)}\,, & (p+p')^2=r^{(1)}\,, \\ & (2k)^2=s\,. \end{array} \tag{2.3}$$

Between them the following identities hold:

$$\begin{aligned} m_A^2+m_B^2+v+w&=s+t^{(0)}+t^{(1)}\,,\\ m_A^2+m_B^2+v'+w'&=s+t^{(0)\prime}+t^{(1)\prime}\,,\\ v+w+v'+w'&=s+r^{(0)}+r^{(1)}\,. \end{aligned} \tag{2.4}$$

Since f, g, and G have both t-channel and u-channel contributions, we write them as

$$\begin{aligned} f(p,q)&=f^{(0)}(p,q)+f^{(1)}(p,q)\,,\\ g(p,q)&=g^{(0)}(p,q)+g^{(1)}(p,q)\,,\\ G(p,p')&=G^{(0)}(p,p')+G^{(1)}(p,p')\,. \end{aligned} \tag{2.5}$$

Now, the following integral representations for $g^{(j)}$ and $G^{(j)}$ [8] can be proven in every order of perturbation theory:

$$g^{(j)}(p,q)=(\pi^2i)^{-1}\int_0^1 dz_0\int_0^1 dz_1\int_0^1 dz_2\int_{-\infty}^{\infty} d\gamma \times\frac{\chi^{(j)}(z_0,z_1,z_2,\gamma)}{(\gamma-z_1v-z_2w-z_0t^{(j)}-i\epsilon)^2}\,, \tag{2.6}$$

$$G^{(j)}(p,p')=(\pi^2i)^{-1}\int_0^1 dy_0\cdots\int_0^1 dy_4\int_{-\infty}^{\infty} d\beta \times\frac{\psi^{(j)}(y_0,\cdots,y_4,\beta)}{(\beta-y_1v'-y_2w'-y_3v-y_4w-y_0r^{(j)}-i\epsilon)^2}\,, \tag{2.7}$$

provided that the theory is renormalizable. Hence it is very natural to assume that $f^{(j)}$ and f_l have the following integral representations:

$$f^{(j)}(p,q)=(\pi^2i)^{-1}\int_0^1 dz_0\int_0^1 dz_1\int_0^1 dz_2\int_{-\infty}^{\infty} d\gamma \times\frac{\varphi^{(j)}(z_0,z_1,z_2,\gamma)}{(\gamma-z_1v-z_2w-z_0t^{(j)}-i\epsilon)^2}\,, \tag{2.8}$$

$$f_l(p)=(\pi^2i)^{-1}\int_0^1 dz_1\int_0^1 dz_2\int_{-\infty}^{\infty} d\gamma \times\frac{\varphi_l(z_1,z_2,\gamma)}{(\gamma-z_1v-z_2w-i\epsilon)^{l+2}}\,. \tag{2.9}$$

[8] The superscript j always goes over 0 and 1 throughout this paper.

In (2.6)–(2.9) the weight functions $\chi^{(j)}$, $\psi^{(j)}$, $\varphi^{(j)}$, and φ_l contain

$$\delta(1-\sum_{i=0}^{2} z_i)\,,\quad \delta(1-\sum_{i=0}^{4} y_i)\,,\quad \delta(1-\sum_{i=0}^{2} z_i)\,,$$

and $\delta(1-z_1-z_2)$ as a factor, respectively. We have omitted to write s explicitly in the arguments of the weight functions.

3. INTEGRAL EQUATIONS FOR THE WEIGHT FUNCTIONS

We consider (2.2) first. Substituting (2.9) and (2.7) with (2.5) in (2.2), we can carry out the Feynman integration. After some calculation, (2.2) becomes as follows[9]:

$$\begin{aligned} &\int_0^1 dz_1\int_0^1 dz_2\int_{-\infty}^{\infty} d\gamma\frac{\varphi_l(z_1,z_2,\gamma)}{(\gamma-z_1v-z_2w-i\epsilon)^{l+2}}\\ &=\int_0^1 dz_1\int_0^1 dz_2\int_{-\infty}^{\infty} d\gamma\int_0^1 dz_1'\int_0^1 dz_2'\int_{-\infty}^{\infty} d\gamma'\\ &\quad\times\frac{K_l(z_1,z_2,\gamma\,;z_1',z_2',\gamma')\,\varphi_l(z_1',z_2',\gamma')}{(\gamma-z_1v-z_2w-i\epsilon)^{l+2}}\,, \end{aligned} \tag{3.1}$$

where

$$K_l\equiv\tilde{K}_l^{(0)}+\tilde{K}_l^{(1)}\,, \tag{3.2}$$

and

$$\begin{aligned} \tilde{K}_l^{(j)}\equiv{}&(-1)^{jl}\int_0^1 dy_0\cdots\int_0^1 dy_4\int_{-\infty}^{\infty} d\beta\,\psi^{(j)}(y_0,\cdots,y_4,\beta)y_0^l\\ &\times\int_0^1 dx(1-x)^{l+1}(a\xi)^{-l-2}\\ &\times\delta(z_1-\xi^{-1}(y_3+a^{-1}y_0c^{(j)}))\\ &\times\delta(z_2-\xi^{-1}(y_4+a^{-1}y_0c^{(1-j)}))\\ &\times\delta(x\gamma-\xi^{-1}[x\beta+(1-x)\gamma'-a^{-1}c^{(0)}c^{(1)}s])\,, \end{aligned} \tag{3.3}$$

with

$$c^{(0)}\equiv xy_1+(1-x)z_1'\,, \tag{3.4a}$$

$$c^{(1)}\equiv xy_2+(1-x)z_2'\,, \tag{3.4b}$$

$$a\equiv c^{(0)}+c^{(1)}+xy_0\,, \tag{3.4c}$$

$$\xi\equiv y_3+y_4+a^{-1}y_0(c^{(0)}+c^{(1)})\,. \tag{3.4d}$$

Now, we assume that we can prove the normal nonforward dispersion relation for the process $C+\bar{C}\to D+\bar{D}$ in every order of perturbation theory. Then it is obvious that $\psi^{(j)}$ vanishes unless

$$\beta\geq(y_1+y_3)m_C^2+(y_2+y_4)m_D^2+y_0\tilde{t}_0^{(j)} \tag{3.5}$$

if

$$\tilde{s}_0\geq s\geq 2(m_C^2+m_D^2)-\tilde{t}_0^{(0)}-\tilde{t}_0^{(1)}\,. \tag{3.6}$$

[9] Equation (3.1) is an analytic relation with respect to s. Hence we can analytically continue it with respect to s.

Here $\tilde{s}_0$, $\tilde{t}_0^{(0)}\equiv\tilde{t}_0$, and $\tilde{t}_0^{(1)}\equiv\tilde{u}_0$ are the lowest normal thresholds[10] for the channels $C+D$, $C+\bar{C}$, and $C+\bar{D}$, respectively, in $G(p,p')$. We further assume

$$s\leq(m_C+m_D)^2,\quad \tilde{t}_0^{(1)}\geq(m_C-m_D)^2. \tag{3.7}$$

Then, as will be shown in the next section, $\varphi_l(z_1,z_2,\gamma)$ vanishes unless

$$\gamma\geq z_1m_C^2+z_2m_D^2. \tag{3.8}$$

We can, therefore, apply to (3.1) the uniqueness theorem of the perturbation-theoretical integral representation.[11] Thus we obtain

$$\varphi_l(z_1,z_2,\gamma)=\int_0^1 dz_1'\int_0^1 dz_2'\int_0^\infty d\gamma' \times K_l(z_1,z_2,\gamma;z_1',z_2',\gamma')\varphi_l(z_1',z_2',\gamma'). \tag{3.9}$$

Our next task is the analytic continuation of (3.9) to the complex l plane. First, we define

$$K_l^{(j)}\equiv(-1)^{jl}\tilde{K}_l^{(j)}. \tag{3.10}$$

Then we have

$$\begin{aligned} K_l&=K_l^{(0)}+K_l^{(1)} \quad \text{for } l \text{ even},\\ K_l&=K_l^{(0)}-K_l^{(1)} \quad \text{for } l \text{ odd}. \end{aligned} \tag{3.11}$$

Let $K_l^{(+)}$ and $K_l^{(-)}$ be the analytic continuations of K_l from l even and from l odd, respectively. As for uniqueness, Carlson's theorem[12] can be applied to $K_l^{(+)}$ and $K_l^{(-)}$ if

$$0\leq\eta\equiv y_0(1-x)a^{-1}\xi^{-1}<1. \tag{3.12}$$

From (3.4) with $z_1'+z_2'=1$, it is evident that $0\leq\eta\leq 1$. Since the contribution from $x\simeq 0$ or 1 is infinitesimal, we may assume $0<x<1$. Then $\eta=1$ is attained only if

$$y_1=y_2=y_3=y_4=0, \tag{3.13}$$

but this case gives no contribution to the integral (3.3), provided that self-energy parts are renormalized. Thus the analytic continuation of (3.9) is always unique.

We should remark, however, that the continued equation is different from the original one for some integers l if there is a finite contribution from the case $\eta=0$, i.e.,

$$y_0=0. \tag{3.14}$$

If we take a graph which has a one-particle intermediate state in the s channel or its reduced graph, the contribution from (3.14) is finite because $\psi^{(j)}$ has a δ function or its derivatives at $y_0=0$.

Now, we return to (2.1). We assume that we can prove the normal nonforward dispersion relation for the process $A+\bar{C}\to\bar{B}+D$ in every order of perturbation theory. Then it is evident that $\chi^{(j)}$ vanishes unless

$$\gamma\geq z_1m_C^2+z_2m_D^2+z_0t_0^{(j)} \tag{3.15}$$

if

$$s_0\geq s\geq m_A^2+m_B^2+m_C^2+m_D^2-t_0^{(0)}-t_0^{(1)}. \tag{3.16}$$

Here s_0, $t_0^{(0)}\equiv t_0$, and $t_0^{(1)}\equiv u_0$ are the lowest normal thresholds for the channels $A+B$, $A+\bar{C}$, and $A+\bar{D}$, respectively, in $g(p,q)$. Let s lie in the intersection of (3.6) and (3.16). We further impose certain conditions on m_A, m_B, m_C, m_D, $t_0^{(0)}$, $t_0^{(1)}$, and $\tilde{t}_0^{(1)}$. Then, as will be shown in the next section, $\varphi^{(j)}$ vanishes unless (3.15) holds. Hence we can apply the uniqueness theorem of the perturbation-theoretical integral representation.[13] Thus (2.1) together with (2.3)–(2.8) yields the following integral equations after the standard calculation:

$$\varphi^{(j)}(z_0,z_1,z_2,\gamma)=\chi^{(j)}(z_0,z_1,z_2,\gamma)+\sum_{k=0}^{1}\int_0^1 dz_0'\int_0^1 dz_1'\int_0^1 dz_2'\int_0^\infty d\gamma' \times K^{(jk)}(z_0,z_1,z_2,\gamma;z_0',z_1',z_2',\gamma') \times\varphi^{(k)}(z_0',z_1',z_2',\gamma'). \tag{3.17}$$

Here

$$\begin{aligned} K^{(j0)}\equiv&\int_0^1 dy_0\cdots\int_0^1 dy_4\int_0^\infty d\beta\,\psi^{(j)}(y_0,\cdots,y_4,\beta)\\ &\times\int_0^1 dx(1-x)(\tilde{a}\xi)^{-2}\delta(z_0-\xi^{-1}\tilde{a}^{-1}y_0(1-x)z_0')\\ &\times\delta(z_1-\xi^{-1}(y_3+\tilde{a}^{-1}y_0c^{(j)}))\\ &\times\delta(z_2-\xi^{-1}(y_4+\tilde{a}^{-1}y_0c^{(1-j)}))\\ &\times\delta(x\gamma-\xi^{-1}[x\beta+(1-x)\gamma'\\ &-\tilde{a}^{-1}(1-x)z_0'(c^{(0)}m_A^2+c^{(1)}m_B^2)\\ &-\tilde{a}^{-1}c^{(0)}c^{(1)}s]), \end{aligned} \tag{3.18}$$

with

$$\tilde{a}\equiv c^{(0)}+c^{(1)}+xy_0+(1-x)z_0', \tag{3.19a}$$

$$\xi\equiv y_3+y_4+\tilde{a}^{-1}y_0[c^{(0)}+c^{(1)}+(1-x)z_0'], \tag{3.19b}$$

and $K^{(j1)}$ can be obtained from $K^{(1-j,0)}$ by interchanging m_A and m_B in the last δ function of (3.18). Since

$$0\leq\tilde{\eta}\equiv\xi^{-1}\tilde{a}^{-1}y_0(1-x)\leq 1, \tag{3.20}$$

$K^{(jk)}$ vanishes unless $z_0\leq z_0'$.

4. SUPPORT PROPERTY

In this section, we investigate under what conditions the support property (3.15) of $\varphi^{(j)}$ is consistent with

[10] In general, $\tilde{s}_0$, $\tilde{t}_0^{(0)}$, and $\tilde{t}_0^{(1)}$ may be some positive constants satisfying (3.5) with (3.6).

[11] N. Nakanishi, Phys. Rev. **127**, 1380 (1962).

[12] E. C. Titchmarsh, *Theory of Functions* (Oxford University Press, New York, 1939), 2nd ed.

[13] In order to prove the uniqueness in this case, we analytically continue v and w to $v<m_C^2$ and $w<m_D^2$. Then our representation can be regarded as a usual dispersion relation for t. Hence the uniqueness of the dispersion relation leads to the property that if $f\equiv f^{(0)}+f^{(1)}\equiv 0$ then $f^{(0)}\equiv f^{(1)}\equiv 0$. Then the uniqueness theorem of Ref. 11 yields the desired result.

(3.17). Namely, our purpose is to show the inequality

$$\gamma \geq z_1 m_C^2 + z_2 m_D^2 + z_0 t_0^{(j)}, \tag{4.1}$$

where

$$\begin{aligned}\gamma = x^{-1}\xi^{-1}[x\beta + (1-x)\gamma' - \tilde{\alpha}^{-1}(1-x)z_0' \\ \times (c^{(0)}m_A^2 + c^{(1)}m_B^2) - \tilde{\alpha}^{-1}c^{(0)}c^{(1)}s]\\ z_0 = \xi^{-1}\tilde{\alpha}^{-1}y_0(1-x)z_0', \\ z_1 = \xi^{-1}(y_3 + \tilde{\alpha}^{-1}y_0 c^{(j)}), \\ z_2 = \xi^{-1}(y_4 + \tilde{\alpha}^{-1}y_0 c^{(1-j)}),\end{aligned} \tag{4.2}$$

under the assumptions (3.5) and

$$\gamma' \geq z_1' m_C^2 + z_2' m_D^2 + z_0' t_0^{(0)}, \tag{4.3}$$

with certain conditions which should be found. On account of (4.2), (3.5), and (4.3), the inequality (4.1) can be rewritten as

$$\begin{aligned}\tilde{\alpha}[c^{(0)}m_C^2 + c^{(1)}m_D^2 + xy_0\tilde{t}_0^{(j)} + (1-x)z_0' t_0^{(0)}] \\ - (1-x)z_0'(c^{(0)}m_A^2 + c^{(1)}m_B^2) - c^{(0)}c^{(1)}s \\ - xy_0(c^{(j)}m_C^2 + c^{(1-j)}m_D^2) - xy_0(1-x)z_0' t_0^{(j)} \geq 0,\end{aligned} \tag{4.4}$$

by making use of (3.4a) and (3.4b).

It is convenient to put

$$\begin{aligned}c^{(0)} = x_1, \quad c^{(1)} = x_2, \\ (1-x)z_0' = x_3, \quad xy_0 = x_4.\end{aligned} \tag{4.5}$$

Then, on account of (3.19a), (4.4) becomes[14]

$$\begin{aligned}(x_1+x_2+x_3+x_4)(x_1 m_C^2 + x_2 m_D^2 + x_3 t_0 + x_4 \tilde{t}_0) \\ - x_1x_3m_A^2 - x_2x_3m_B^2 - x_1x_4m_C^2 - x_2x_4m_D^2 \\ - x_1x_2 s - x_3x_4 t_0 \geq 0\end{aligned} \tag{4.6}$$

for $j=0$, and

$$\begin{aligned}(x_1+x_2+x_3+x_4)(x_1 m_C^2 + x_2 m_D^2 + x_3 t_0 + x_4 \tilde{u}_0) \\ - x_1x_3m_A^2 - x_2x_3m_B^2 - x_2x_4m_C^2 - x_1x_4m_D^2 \\ - x_1x_2 s - x_3x_4 u_0 \geq 0\end{aligned} \tag{4.7}$$

for $j=1$. It is evident that (4.6) is satisfied if

$$\begin{aligned}(x_1+x_2+x_3)(x_1 m_C^2 + x_2 m_D^2 + x_3 t_0) - x_1x_3 m_A^2 \\ - x_2x_3 m_B^2 - x_1x_2 s \geq 0.\end{aligned} \tag{4.8}$$

Since (4.8) is a special case of (4.7), it is sufficient to consider (4.7) alone. The left-hand side of (4.7) is exactly the same with the denominator function V of the Feynman parameteric integral corresponding to Fig. 1 with internal masses $m_1 = m_C$, $m_2 = m_D$, $m_3 = t_0^{1/2}$, $m_4 = \tilde{u}_0^{1/2}$, and with invariant squares s and u_0. The conditions for $V \geq 0$ are well known.[15]

In a quite analogous way, we can investigate the conditions for self-reproducing the support property in the remaining two integrals of (3.17). The result is

$$\begin{aligned}(x_1+x_2+x_3+x_4)(x_1 m_C^2 + x_2 m_D^2 + x_3 u_0 + x_4 \tilde{u}_0) \\ - x_1x_3m_B^2 - x_2x_3m_A^2 - x_2x_4m_C^2 - x_1x_4m_D^2 \\ - x_1x_2 s - x_3x_4 t_0 \geq 0.\end{aligned} \tag{4.9}$$

[14] Remember $t_0^{(0)} \equiv t_0$, $t_0^{(1)} \equiv u_0$, etc.

[15] R. Karplus, C. M. Sommerfield, and E. H. Wichmann, Phys. Rev. 114, 376 (1959).

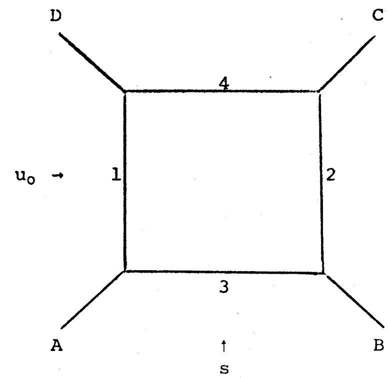

FIG. 1. Graphical representation of the inequality (4.7).

It will be convenient to write down sufficient conditions for (4.7) and (4.9) more explicitly. First of all, we need the stability conditions

$$t_0 \geq \max[(m_A - m_C)^2, (m_B - m_D)^2], \tag{4.10a}$$

$$u_0 \geq \max[(m_B - m_C)^2, (m_A - m_D)^2], \tag{4.10b}$$

$$\tilde{u}_0 \geq (m_C - m_D)^2. \tag{4.10c}$$

If we require the normal thresholds:

$$s \leq (m_C + m_D)^2, \tag{4.11a}$$

$$u_0 \leq (t_0^{1/2} + \tilde{u}_0^{1/2})^2, \tag{4.11b}$$

$$t_0 \leq (u_0^{1/2} + \tilde{u}_0^{1/2})^2, \tag{4.11c}$$

then (4.11a), (4.11b), and (4.11c) require the following additional conditions, (*a*), (*b*), and (*c*), respectively, for the absence of anomalous thresholds[15]:

$$\begin{aligned}&(a) \quad m_C(t_0 + m_D^2 - m_B^2) + m_D(t_0 + m_C^2 - m_A^2) \geq 0, \\ &(b) \quad \tilde{u}_0^{1/2}(t_0 + m_C^2 - m_A^2) + t_0^{1/2}(\tilde{u}_0 + m_C^2 - m_D^2) \geq 0, \\ &\qquad \tilde{u}_0^{1/2}(t_0 + m_D^2 - m_B^2) + t_0^{1/2}(\tilde{u}_0 + m_D^2 - m_C^2) \geq 0, \\ &(c) \quad \tilde{u}_0^{1/2}(u_0 + m_C^2 - m_B^2) + u_0^{1/2}(\tilde{u}_0 + m_C^2 - m_D^2) \geq 0, \\ &\qquad \tilde{u}_0^{1/2}(u_0 + m_D^2 - m_A^2) + u_0^{1/2}(\tilde{u}_0 + m_D^2 - m_C^2) \geq 0.\end{aligned} \tag{4.12}$$

If we take more stringent conditions

$$\begin{aligned}s \leq (m_C - m_D)^2, \\ u_0 \leq (t_0^{1/2} - \tilde{u}_0^{1/2})^2, \\ t_0 \leq (u_0^{1/2} - \tilde{u}_0^{1/2})^2,\end{aligned} \tag{4.13}$$

instead of (4.11), then no additional conditions are necessary. It is noteworthy that there is no restriction for $\tilde{t}_0$.

Finally, we consider the Bethe-Salpeter equation (3.9). This corresponds to a special case $x_3 = 0$ of the above. Therefore, we obtain only two conditions (4.10c) and (4.11a), which have been written in (3.7).

We have thus proved the *consistency* of the support properties (3.8) for φ_l and (3.15) for $\varphi^{(j)}$ under the conditions (3.7) and either (4.10)–(4.12) or (4.10) with (4.13), respectively. Almost all practically important cases satisfy the required conditions.

Rigorously speaking, the above reasoning cannot exclude the possible existence of weight functions which fail to vanish outside of (3.8) or (3.15) even if all the above conditions are satisfied. But the existence of such "bad" solutions is extremely unlikely because of the following reasons. As for the Bethe-Salpeter equation, we know that it has no such bad solutions in the ladder approximation.[16] As for the inhomogeneous equation for the scattering amplitude, the iterative solution, if it exists, differs from a bad solution. It will be very reasonable to assume the nonexistence of bad solutions.

5. HIGH-ENERGY BEHAVIOR

In order to discuss the high-energy behavior of the scattering amplitude in the t channel, it is convenient to introduce the even and odd amplitudes:

$$\begin{aligned}\varphi^{(+)}&\equiv\varphi^{(0)}+\varphi^{(1)},\\ \varphi^{(-)}&\equiv\varphi^{(0)}-\varphi^{(1)}.\end{aligned}\tag{5.1}$$

The high-energy behavior of the amplitude f in the t channel is determined by the behavior of $\varphi^{(\pm)}$ at a neighborhood of $z_0=0$. When $z_0\simeq z_0'\simeq 0$, we can neglect higher order terms of z_0 and z_0' in (3.18). In this way, we have

$$\begin{aligned}K^{(00)}&\simeq K^{(11)}\simeq K^{(0)},\\ K^{(10)}&\simeq K^{(01)}\simeq K^{(1)},\end{aligned}\tag{5.2}$$

where

$$\begin{aligned}K^{(j)}\equiv&\int_0^1 dy_0\cdots\int_0^1 dy_4\int_0^\infty d\beta\,\psi^{(j)}(y_0,\cdots,y_4,\beta)\\ &\times\int_0^1 dx(1-x)(a\xi)^{-2}\delta(z_0-\xi^{-1}a^{-1}y_0(1-x)z_0')\\ &\times\delta(z_1-\xi^{-1}(y_3+a^{-1}y_0c^{(j)}))\\ &\times\delta(z_2-\xi^{-1}(y_4+a^{-1}y_0c^{(1-j)}))\\ &\times\delta(x\gamma-\xi^{-1}[x\beta+(1-x)\gamma'-a^{-1}c^{(0)}c^{(1)}s]).\end{aligned}\tag{5.3}$$

We assume that $\chi^{(j)}$ and $\psi^{(j)}$ are bounded at $z_0\simeq 0$ and at $y_0\simeq 0$, respectively,[17] and that $\varphi^{(\pm)}$ is *not* bounded at $z_0\simeq 0$. Then we obtain the "asymptotic equations":

$$\varphi^{(+)}\simeq(K^{(0)}+K^{(1)})\varphi^{(+)},\tag{5.4a}$$

$$\varphi^{(-)}\simeq(K^{(0)}-K^{(1)})\varphi^{(-)}.\tag{5.4b}$$

Here we have employed the operator notation for simplicity.

Now, we can write

$$\varphi^{(+)}(z_0,z_1,z_2,\gamma)\simeq F^{(+)}(z_0)\tilde{\varphi}^{(+)}(z_1,z_2,\gamma),\tag{5.5}$$

where $\tilde{\varphi}^{(+)}$ contains $\delta(1-z_1-z_2)$ as a factor. We substitute (5.5) in (5.4a) with (5.3). Then the z_0'-integration is simply

$$\int_0^1 dz_0'\delta(z_0-\eta z_0')F^{(+)}(z_0')=\eta^{-1}F^{(+)}(z_0/\eta)\theta(\eta-z_0),\tag{5.6}$$

where η is given by (3.12). Therefore (5.4a) can be satisfied if

$$F^{(+)}(z_0)=\text{const Pf }z_0^{-l-1}(\ln 1/z_0)^n(\ln\ln 1/z_0)^r\cdots,\tag{5.7}$$

where l, n, r, etc. may depend only on s. In (5.7) the symbol Pf[18] indicates that when one carries out an integration over z_0 one should first calculate the integral in $\mathrm{Re}\,l<0$, $\mathrm{Re}\,n>-1$, etc., and then analytically continue the result with respect to l, n, etc. The integration over y_0 in (5.3) cannot change the degree of the singularity of (5.6) at $z_0=0$ as far as $\mathrm{Re}\,l>-1$.

On the other hand, from (5.6) and (5.7) we can easily see that $\tilde{\varphi}^{(+)}(z_1,z_2,\gamma)$ in (5.5) must satisfy a homogeneous integral equation, which is exactly the same with the partial-wave Bethe-Salpeter equation continued from l even, i.e.,

$$\tilde{\varphi}^{(+)}=K_l^{(+)}\tilde{\varphi}^{(+)}.\tag{5.8}$$

Thus $\tilde{\varphi}^{(+)}$ is equal to a solution φ_l of (3.9) continued from l even apart from a normalization factor which depends only on s. The eigenvalue problem (5.8) determines l as a function of s.

Quite similarly, we have

$$\varphi^{(-)}(z_0,z_1,z_2,\gamma)\simeq F^{(-)}(z_0)\tilde{\varphi}^{(-)}(z_1,z_2,\gamma),\tag{5.9}$$

$$F^{(-)}(z_0)=\text{const Pf }z_0^{-l'-1}(\ln 1/z_0)^{n'}\cdots,\tag{5.10}$$

$$\tilde{\varphi}^{(-)}=K_{l'}^{(-)}\tilde{\varphi}^{(-)}.\tag{5.11}$$

We have thus established quite generally that the high-energy behavior of the scattering amplitude is closely related to the continued partial-wave Bethe-Salpeter equation. The connection with the Regge analysis was already discussed in I. While l in (5.7) gives the trajectory of the singularity in the complex angular momentum plane, the other parameters concern the nature (or strength) of the singularity. The high-energy limit corresponding to (5.7) is proportional to

$$(-t)^l[\ln(-t)]^n[\ln\ln(-t)]^r\cdots.\tag{5.12}$$

The factorizability[19] of the "residue" functions $\tilde{\varphi}^{(\pm)}$ can be shown by exactly the same way as the reasoning of Amati, Stanghellini, and Fubini.[3] It is somewhat remarkable that the factorizability is still valid even if we have a branch cut in the complex l plane. But it should be noticed that this reasoning[3] applies only when the

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[16] R. E. Cutkosky, Phys. Rev. **96**, 1135 (1954). N. Nakanishi, *ibid.* **130**, 1230 (1963).

[17] If $\psi^{(j)}$ contains a term proportional to $\delta^{(l)}(y_0)$, that part always gives $\delta^{(l)}(z_0)$ irrelevantly to the input function $\varphi^{(\pm)}$ in the right-hand side of the asymptotic equation. $\delta^{(l)}(y_0)$ corresponds to an "elementary" particle having spin l in the s channel.

[18] L. Schwartz, *Théorie des Distributions* (Hermann & Cie., Paris, 1950), Chap. II.

[19] M. Gell-Mann, Phys. Rev. Letters **8**, 263 (1962); V. N. Gribov and I. Ya. Pomeranchuk, *ibid.* **8**, 343 (1962); Y. Hara, Progr. Theoret. Phys. (Kyoto) **28**, 711 (1962).

particles have the same quantum numbers except for their masses.

Finally, we will remark the form of $F^{(\pm)}(z_0)$. Strictly speaking, $F^{(\pm)}(z_0)$ may be a sum of some terms in which the imaginary parts of l (and/or n, etc. $\cdots$) only are different from one another. But since such a possibility is very unlikely and any work done so far seems not to consider such a case, we assume that $F^{(\pm)}(z_0)$ consists of a single term like (5.7) or (5.10). Then l, n, etc., must be real because the weight function is a real function.

6. FURTHER CONSIDERATION ON HIGH-ENERGY LIMIT

In the preceding section, we have investigated the high-energy behavior of the Feynman amplitude under the assumption that $\psi^{(j)}$ is bounded at $y_0 \simeq 0$. But since the kernel $G(p,p')$ contains the contribution from infinitely many graphs, this assumption is not always assured and $\psi^{(j)}$ may behave just like $F^{(\pm)}(y_0)$ at $y_0 \simeq 0$. Hence we write

$$\psi^{(j)}(y_0,\cdots,y_4,\beta) \simeq H^{(j)}(y_0)\tilde{\psi}^{(j)}(y_1,\cdots,y_4,\beta) \qquad (6.1)$$

and, for simplicity, assume

$$F^{(+)}(z_0) = \mathrm{Pf}\, z_0^{-l-1}(\ln 1/z_0)^n, \qquad (\mathrm{Re}\, l > -1), \qquad (6.2a)$$

$$H^{(0)}(y_0) = \mathrm{Pf}\, y_0^{-k-1}(\ln 1/y_0)^r, \qquad (\mathrm{Re}\, k \geq -1), \qquad (6.2b)$$

and likewise for $F^{(-)}$ and $H^{(1)}$. Then, according to (5.3), the asymptotic behavior of $K^{(0)}\varphi^{(+)}$, say, is determined by the following integral:

$$I \equiv \int_0^1 dy_0 \int_0^1 dz_0' H^{(0)}(y_0)F^{(+)}(z_0')\delta(z_0 - \kappa y_0 z_0'), \qquad (6.3)$$

with

$$\kappa \equiv (1-x)(c^{(0)}+c^{(1)})^{-1}(y_3+y_4)^{-1} \simeq \eta/y_0. \qquad (6.4)$$

The evaluation of (6.3) is given in the Appendix. The result is as follows.

(a) $l > k$:

$$I \simeq \Gamma(r+1)(l-k)^{-r-1}\theta(\kappa - z_0) \times \kappa^l z_0^{-l-1}(\ln \kappa/z_0)^n. \qquad (6.5)$$

(b) $l = k$:

$$I = B(n+1, r+1)\theta(\kappa - z_0) \times \kappa^l z_0^{-l-1}(\ln \kappa/z_0)^{n+r+1}. \qquad (6.6)$$

(c) $l < k$:

$$I \simeq \Gamma(n+1)(k-l)^{-n-1}\theta(\kappa - z_0) \times \kappa^k z_0^{-k-1}(\ln \kappa/z_0)^r. \qquad (6.7)$$

In the above, B and Γ denote Euler's beta and gamma functions, respectively.

As is seen from (6.5), in the case $l > k$, the asymptotic behavior is not affected by the presence of the singularity of $\psi^{(0)}$. Therefore, (5.4a) is still valid in this case, provided that $\psi^{(1)}$ is of similar nature. In the case $l = k$, if $\mathrm{Re}\, r > -1$, the singularity given by (6.6) is stronger than the original one (6.2a). In this case, therefore, $F^{(+)}(z_0)$ is no longer self-reproducing. Finally, in the case $l < k$, the singularity given by (6.7) is independent of $F^{(+)}(z_0)$.

Furthermore, if $\chi^{(j)}$ is not less singular than $\varphi^{(\pm)}$, we must, of course, take account of the inhomogeneous term. In this case, it is preferable to rewrite (2.1) by taking another intermediate state instead of $C+D$.

7. NONLINEAR INTEGRAL EQUATION

A homogeneous equation like (5.8) or (5.11) cannot determine the normalization of the residue function $\bar{\varphi}^{(\pm)}$. Amati, Stanghellini, and Fubini[3] have proposed to use a nonlinear integral equation for determining the normalization. In this section we shall generalize this method.

We consider the *elastic scattering only* ["elastic" is referred to in the s (or u) channel, i.e., $A=C$, $B=D$]. Then our basic equation (2.1) becomes

$$f = G + Gf \qquad (7.1)$$

in the operator notation. Now, we replace G by λG:

$$f = \lambda G + \lambda G f, \qquad (7.2)$$

where λ is a parameter. Then the solution f of (7.2) is, of course, a function of λ, and all arguments made in the previous sections equally apply to the new f. We introduce a function $\bar{f} \equiv \partial f/\partial\lambda$, which satisfies the following equation:

$$\bar{f} = G + Gf + \lambda G\bar{f}, \qquad (7.3)$$

or,

$$\bar{f} = \lambda^{-1} f + \lambda G \bar{f}. \qquad (7.4)$$

If an operator $(1-\lambda G)^{-1}$ is well defined,[20] (7.3) with (7.2) leads to

$$\bar{f} = (1-\lambda G)^{-1}(G+Gf) = (1-\lambda G)^{-1}G(1+f) = \lambda^{-1} f(1+f), \qquad (7.5)$$

namely, we have a nonlinear integral equation

$$\lambda \bar{f}(p,q) = f(p,q) + \int d^4p' f(p,p')f(p',q). \qquad (7.6)$$

We assume that the asymptotic limit of $\bar{f}$ is obtained from that of f by operating $\partial/\partial\lambda$, namely, the less singular part of $\varphi^{(\pm)}$ at $z_0 \simeq 0$ does not contribute to the leading part of $\bar{\varphi}^{(\pm)} \equiv \partial\varphi^{(\pm)}/\partial\lambda$. This assumption was tacitly made by Amati *et al.*[3] A counter example may be provided if the less singular part contains a term containing such a factor as $\sin(\lambda z_0^{-m})$, $m > 0$. But such an oscillatory behavior will be inconsistent with (7.6).

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[20] If $|\lambda|$ is very small, this operator will probably exist. After the calculation (7.5), we should analytically continue (7.6) with respect to λ.

If the above assumption is accepted, the leading singularities of $\bar{\varphi}^{(\pm)}$ are given by

$$\bar{\varphi}^{(+)}\simeq\varphi^{(+)}\ln(1/z_0)\partial l/\partial\lambda\,,\qquad \bar{\varphi}^{(-)}\simeq\varphi^{(-)}\ln(1/z_0)\partial l'/\partial\lambda\,. \tag{7.7}$$

Thus the singularities of $\bar{\varphi}^{(\pm)}$ are always stronger than those of $\varphi^{(\pm)}$ as far as $\partial l/\partial\lambda\neq0$ and $\partial l'/\partial\lambda\neq0$, respectively. As is seen from (7.4), therefore, the asymptotic equations for $\bar{\varphi}^{(\pm)}$ are identical with (5.4) apart from λ.[21]

Now, we consider the nonlinear integral equation (7.6). In addition to (2.8), we employ the following integral representation for $f(p,p')$:

$$f^{(j)}(p,p')=(\pi^2 i)^{-1}\int_0^1 dy_0\cdots\int_0^1 dy_4\int_0^\infty d\beta\times\frac{\hat{\varphi}^{(j)}(y_0,\cdots,y_4,\beta)}{(\beta-y_1v'-y_2w'-y_3v-y_4w-y_0r^{(j)}-i\epsilon)^2}\,, \tag{7.8}$$

$\hat{\varphi}^{(j)}$ being related to $\varphi^{(j)}$ through

$$\varphi^{(j)}(z_0,z_1,z_2,\gamma)=\int_0^1 dy_0\cdots\int_0^1 dy_4\int_0^\infty d\beta\, y^{-2}\delta(z_0-y_0y^{-1})\delta(z_1-y_3y^{-1})\times\delta(z_2-y_4y^{-1})\delta(\gamma-y^{-1}(\beta-y_1m_A{}^2-y_2m_B{}^2))\times\hat{\varphi}^{(j)}(y_0,\cdots,y_4,\beta)\,, \tag{7.9}$$

with $y\equiv y_0+y_3+y_4$. Then the integral equations for the weight functions are obtained completely analogously. On account of (7.7), the asymptotic equations become

$$\bar{\varphi}^{(+)}\simeq(\hat{K}^{(0)}+\hat{K}^{(1)})\varphi^{(+)}\,, \tag{7.10a}$$

$$\bar{\varphi}^{(-)}\simeq(\hat{K}^{(0)}-\hat{K}^{(1)})\varphi^{(-)}\,, \tag{7.10b}$$

where $\hat{K}^{(j)}$ is obtained from $K^{(j)}$ by replacing $\psi^{(j)}$ by $\hat{\varphi}^{(j)}$.

For simplicity, we assume the following behaviors:

$$\varphi^{(+)}\sim z_0{}^{-l-1}(\ln 1/z_0)^n\,, \tag{7.11a}$$

$$\varphi^{(-)}\sim z_0{}^{-l'-1}(\ln 1/z_0)^{n'}\,. \tag{7.11b}$$

Then we have

$$\bar{\varphi}^{(+)}\sim z_0{}^{-l-1}(\ln 1/z_0)^{n+1}\,, \tag{7.12a}$$

$$\bar{\varphi}^{(-)}\sim z_0{}^{-l'-1}(\ln 1/z_0)^{n'+1}\,, \tag{7.12b}$$

and

$$\hat{\varphi}^{(+)}\equiv\hat{\varphi}^{(0)}+\hat{\varphi}^{(1)}\sim y_0{}^{-l-1}(\ln 1/y_0)^n\,, \tag{7.13a}$$

$$\hat{\varphi}^{(-)}\equiv\hat{\varphi}^{(0)}-\hat{\varphi}^{(1)}\sim y_0{}^{-l'-1}(\ln 1/y_0)^{n'}\,. \tag{7.13b}$$

We consider the case $l>l'$. The nonlinear terms $\hat{K}^{(j)}\varphi^{(+)}$ and $\hat{K}^{(j)}\varphi^{(-)}$ behave like

$$\hat{K}^{(j)}\varphi^{(+)}\sim z_0{}^{-l-1}(\ln 1/z_0)^{2n+1}\,, \tag{7.14a}$$

$$\hat{K}^{(j)}\varphi^{(-)}\sim z_0{}^{-l-1}(\ln 1/z_0)^{n}\,, \tag{7.14b}$$

according to (6.6) and (6.7), respectively. In order that (7.14b) be consistent with (7.12b) in (7.10b), a cancellation of the leading terms must take place between $\hat{K}^{(0)}\varphi^{(-)}$ and $-\hat{K}^{(1)}\varphi^{(-)}$. Since a similar cancellation may happen also in (7.10a), the comparison between (7.12a) and (7.14a) yields only

$$n\geq0\,. \tag{7.15}$$

This result excludes the possibility that the leading singularity is a bounded branch point in the complex l plane.

In the case $l=l'$, it may be plausible that $\hat{\varphi}^{(0)}$ behaves like

$$y_0{}^{-l-1}(\ln 1/y_0)^n \tag{7.16}$$

and $\hat{\varphi}^{(1)}$ is less singular. This is trivially true in the ladder approximation because then $\hat{\varphi}^{(1)}\equiv0$. In this case we have

$$\hat{K}^{(0)}\varphi^{(\pm)}\sim z_0{}^{-l-1}(\ln 1/z_0)^{2n+1} \tag{7.17}$$

and no cancellation can happen in (7.10). Therefore, (7.12) and (7.17) lead to

$$n=n'=0\,, \tag{7.18}$$

namely, we have the Regge behavior for the scattering amplitude.

If both $\hat{\varphi}^{(0)}$ and $\hat{\varphi}^{(1)}$ behave like (7.16), it will be natural to expect that f has a certain crossing symmetry between the t and the u channel. For simplicity, consider the elastic scattering of two identical neutral particles for which

$$f(p,-p')=f(p,p')=f(-p,p')\,. \tag{7.19}$$

Then the uniqueness theorem of the perturbation-theoretical integral representation[13] leads to

$$\hat{\varphi}^{(1)}(y_0,y_1,y_2,y_3,y_4,\beta)=\hat{\varphi}^{(0)}(y_0,y_2,y_1,y_3,y_4,\beta)\,,\qquad \varphi^{(1)}(z_0,z_1,z_2,\gamma)=\varphi^{(0)}(z_0,z_2,z_1,\gamma)\,. \tag{7.20}$$

Hence (5.3) together with (3.4a) and (3.4b) gives

$$\hat{K}^{(1)}\varphi^{(\pm)}=\pm\hat{K}^{(0)}\varphi^{(\pm)}\,, \tag{7.21}$$

so that (7.10) reduces to

$$\bar{\varphi}^{(\pm)}\simeq2\hat{K}^{(0)}\varphi^{(\pm)}\,,\quad\text{or}\quad \bar{\varphi}^{(j)}\simeq2\hat{K}^{(0)}\varphi^{(j)}\,. \tag{7.22}$$

Hence we obtain again the Regge behavior (7.18).

The above result will lead to the following alternative conjectures.

(a) All elastic scattering amplitudes will exhibit the Regge behavior.

(b) The scattering amplitude of two alike particles will exhibit the Regge behavior, but that of two unlike ones may not.

[21] Therefore, $\bar{\varphi}^{(+)}$ provides a good counterexample to the ansatz of the Regge behavior made by Bertocchi, Fubini, and Tonin (Ref. 1).

The former is more aesthetic, but the recent experimental results of the high-energy proton-proton and pion-proton scatterings[22] are favorable to the latter.

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APPENDIX

We shall derive (6.5)–(6.7) from (6.3). Because of the symmetry of (6.3) with respect to (l,n) and (k,r), it is sufficient to consider the case $l\geq k$ only. Carrying out the integration over z_0', we have

$$I=\theta(\kappa-z_0)\kappa^l z_0^{-l-1}\,\mathrm{Pf}\int_{z_0/\kappa}^{1}dy_0 \times y_0^{l-k-1}(\ln 1/y_0)^r(\ln\kappa y_0/z_0)^n. \quad \text{(A1)}$$

Putting

$$y_0=\exp(-x\ln\kappa/z_0)\,, \quad \text{(A2)}$$

we obtain

$$I=\theta(\kappa-z_0)\kappa^l z_0^{-l-1}(\ln\kappa/z_0)^{n+r+1}J\,, \quad \text{(A3)}$$

where

$$J\equiv\mathrm{Pf}\int_0^1 dx\, x^r(1-x)^n\exp[-x(l-k)\ln\kappa/z_0]. \quad \text{(A4)}$$

In the case $l=k$, (A3) with (A4) immediately leads to (6.6). For $l>k$, the exponential in (A4) becomes extremely small except for $x\simeq 0$ because we are interested in $z_0\simeq 0$. Hence we may approximate J by replacing $(1-x)^n$ by 1.[23] Then we obtain (6.5).

[22] K. J. Foley, S. J. Lindenbaum, W. A. Love, S. Ozaki, J. J. Russell, and L. C. L. Yuan, Phys. Rev. Letters **10**, 376 (1963).

[23] The analytic continuation with respect to n and r does not invalidate this approximation. This can be checked by expanding $(1-x)^n$ into the Taylor series and integrating term by term.

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Analytic Continuation in Complex Angular Momentum and Integral Equations*

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An attack is made on the problem of the analytic continuation in the angular momentum variable l of amplitudes defined by integral equations beyond the value of Rel at which the kernel ceases to be of the Schmidt type and the Fredholm theory cannot be applied. A general technique is developed and applied to the Yukawa potential case and to the ladder graph series in the φ^3 theory. In both cases meromorphy is established for Re$l>-\frac{5}{2}$ and a procedure is indicated for a stepwise continuation to the entire l plane.

1. INTRODUCTION

THE importance of analyticity properties of scattering amplitudes in the complex angular momentum variable has motivated certain field theoretic approximations[1–13] in the framework of simple models. These models consist essentially of series of Feynman graphs studied either on the basis of integral equations[1–5] or by obtaining asymptotic forms[6–13] for large values of the momentum transfer.

The present situation indicates that sets of *planar*[1–11] Feynman graphs lead to [interpolating] partial-wave amplitudes $F(s,l)$ which are *meromorphic* in l at least in some region beyond the analyticity domain specified by the number of subtractions in the momentum transfer t. With the exception of the "superconvergent" φ^3 theory, one obtains in addition fixed branch points, e.g., in the φ^4 case and in the model of scalar particles interacting through the exchange of vector mesons.[3,5,9] These fixed branch points seem to be quite analogous to those appearing in the r^{-2} potential case.

This close analogy with the nonrelativistic potential case is more or less expected because for planar graphs the ρ_{tu} spectral function vanishes as in the case of

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